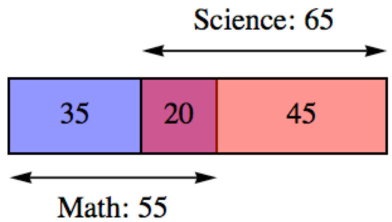
Dual Dig Level I (2012) - Solutions

1. The 100 students at a school must take a math class or a science class this term; some take both. This term, if 55 of the students are taking a math class, and if 20 of the students are taking both math and science classes, how many of the students are taking a science class?

   Explanation: 55 – 20 = 35 students are taking math but not science. The other 100 – 35 = 65 students must all be taking science. The Venn diagram below (using rectangles instead of circles) illustrates this:

   Answer: 65 students

2. What is the average of $\frac{1}{6}$ and $\frac{1}{8}$?

   Explanation: $\frac{\frac{1}{6}+\frac{1}{8}}{2} = \frac{\left(\frac{1}{6}+\frac{1}{8}\right)}{2}\cdot\frac{24}{24} = \frac{4+3}{48} = \frac{7}{48}$, the average of $\frac{8}{48}\left(=\frac{1}{6}\right)$ and $\frac{6}{48}\left(=\frac{1}{8}\right)$.

   Answer: $\frac{7}{48}$

3. Given that $x<0$, simplify: $\sqrt{x^2}+x$

   Explanation: $\sqrt{x^2}+x = |x|+x = (-x)+x = 0$, since $|x| = -x$ if $x<0$. Experimenting with negative values of x can help. For example, for $x=-3$, $\sqrt{(-3)^2}+(-3) = \sqrt{9}-3 = 3-3 = 0$.

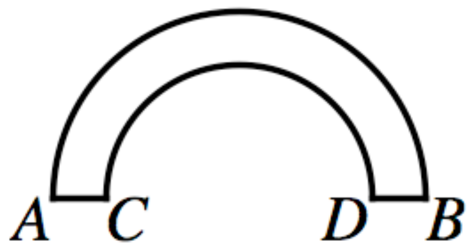
   Answer: 0

4. Rick gives his favorite integer to Larry and Ken. Larry squares Rick's integer, Ken doubles Rick's integer, and they both give their results to Tammi. When Tammi adds the results from Larry and Ken, she gets the cube of Rick's integer. What are the three possibilities for Rick's integer?

   Explanation: Let x = Rick's integer. Solve:
   $x^2+2x = x^3 \iff 0 = x^3-x^2-2x \iff 0 = x\left(x^2-x-2\right) \iff 0 = x(x-2)(x+1)$

   Answer: 0, 2, or -1

5. The figure below represents an audience pit. The boundary of the pit consists of two semicircles and two line segments. The two semicircles are the upper halves of two concentric circles (i.e., circles with the same center). The semicircle $\overset{\frown}{AB}$ is the upper half of a circle of radius 7 yards. The semicircle $\overset{\frown}{CD}$ is the upper half of a circle of radius 5 yards. Find the total perimeter of the boundary of the pit.

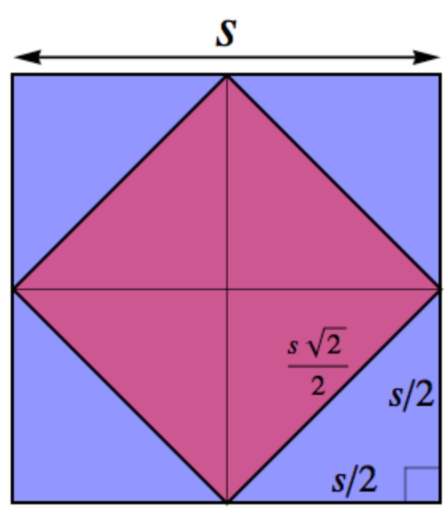


Explanation: The length of semicircle $\overset{\frown}{AB}$ is 7π yards. The length of semicircle $\overset{\frown}{CD}$ is 5π yards. The length of each line segment $\overline{AC}$ and $\overline{BD}$ is 2 yards. The total perimeter is then: $7\pi + 5\pi + 2 + 2 = (12\pi + 4)$ yards

Answer: $(12\pi + 4)$ yards

6. A square is formed by joining the midpoints of the sides of a larger square. The area of the smaller square will then be what fraction of the area of the larger square?

Explanation: The diagram below should help.



If we break the larger square region into quadrants, we see that the smaller square region takes up half of each quadrant. Also, $\dfrac{\text{Area of small square}}{\text{Area of large square}} = \dfrac{\left(\frac{s\sqrt{2}}{2}\right)^2}{s^2} = \dfrac{\left(\frac{s}{\sqrt{2}}\right)^2}{s^2} = \dfrac{\left(\frac{s^2}{2}\right)}{s^2} = \dfrac{1}{2}$.

Answer: $\dfrac{1}{2}$

7. What is the smallest positive integer that is divisible by 2, 3, 4, 5, 6, 7, 8, 9, and 10?

Explanation: LCM(2, 3, 4, 5, 6, 7, 8, 9, 10) = $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 8 \cdot 9 \cdot 5 \cdot 7 = 2520$.
Note that $8 = 2^3$, which covers divisibility by 2, 4, and 8.
Note that $9 = 3^2$, which covers divisibility by 3 and 9.
We need 5 and 7 as factors, since they are prime.
$6 = 2 \cdot 3$, and $10 = 2 \cdot 5$, but those are covered by now.

Answer: 2520

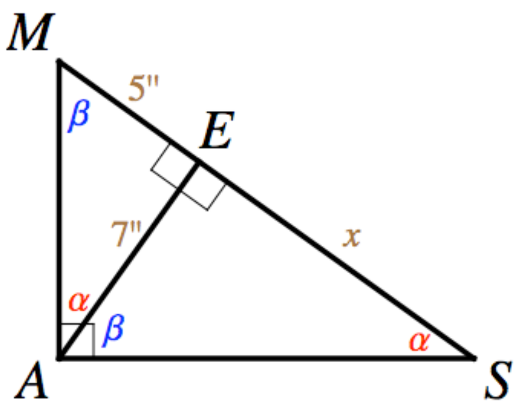
8. Solve for *x*: $\frac{1}{81}\left(9^x\right)^{x+4} = 81^{2x+7}$

Explanation:
Rewrite as: $3^{-4} \bullet \left(3^{2x}\right)^{x+4} = \left(3^4\right)^{2x+7} \Leftrightarrow 3^{2x^2+8x-4} = 3^{8x+28} \Leftrightarrow 2x^2 + 8x - 4 = 8x + 28$.
This simplifies to: $x^2 = 16 \Leftrightarrow x = \pm 4$.

Answer: $\{-4, 4\}$

9. Consider Triangle *MAS* with $\overline{MA} \perp \overline{AS}$, with point *E* somewhere on $\overline{MS}$. A new line segment is drawn from *A* to *E* such that $\overline{AE} \perp \overline{MS}$. Given that *ME* = 5 inches, and *AE* = 7 inches, find the length of $\overline{ES}$.

Explanation:



The sides facing the angles of measure α and β have proportional lengths:

$$\frac{5}{7} = \frac{7}{x} \Leftrightarrow x = \frac{49}{5}$$

Answer: $\frac{49}{5}$ inches, or $9\frac{4}{5}$ inches, or 9.8 inches

10. Find all real solutions of: $\left| x^3 - 8 \right| > 0$

Explanation: Absolute value is nonnegative, so all real numbers will satisfy the inequality, except for $x = 2$. This is because 2 is the sole real solution of $x^3 - 8 = 0$.

Answer: $\left\{ x \in \mathbb{R} \mid x \neq 2 \right\}$, or $(-\infty, 2) \cup (2, \infty)$

11. Find the equation of the parabola in the usual *xy*-plane that contains the points (1, 8), (0, 7), and $(-2, 23)$.

Explanation: Using the equation of a general parabola: $y = ax^2 + bx + c$, and the above ordered pairs, we can write a 3x3 system of equations:

$$\begin{cases} 8 = a + b + c \\ 7 = 0 + 0 + c \\ 23 = 4a - 2b + c \end{cases} \Rightarrow \text{substituting } c = 7 \text{ and simplifying, we get the 2x2 system: } \begin{cases} a + b = 1 \\ 2a - b = 8 \end{cases}$$

From the 2x2 system, we obtain: $a = 3$, and $b = -2$. Substituting back into the original equation for the parabola, we get:

Answer: $y = 3x^2 - 2x + 7$

12. Simplify completely: $\left(\dfrac{-36x^{-3}y^{-7}z^6}{24x^{-4}y^{-5}z^{-2}} \right)^{-2} \left(\dfrac{24x^{-2}y^3z^6}{4z^{12}} \right)^0 \left(\dfrac{3x^5y^4}{z^{-2}} \right)^3$. Assume *x*, *y*, and *z* are nonzero.

Explanation: $\left(\dfrac{-3x^4y^5z^6z^2}{2x^3y^7} \right)^{-2} (1)\left(3x^5y^4z^2\right)^3 \Leftrightarrow \left(\dfrac{2x^3y^7}{-3x^4y^5z^6z^2} \right)^2 \left(27x^{15}y^{12}z^6\right) \Leftrightarrow$

$$\Leftrightarrow \left(\frac{2y^2}{-3xz^8} \right)^2 \left(27x^{15}y^{12}z^6\right) \Leftrightarrow \left(\frac{4y^4}{9x^2z^{16}} \right)\left(27x^{15}y^{12}z^6\right) \Leftrightarrow \frac{12x^{13}y^{16}}{z^{10}}$$

Answer: $\dfrac{12x^{13}y^{16}}{z^{10}}$

13. Determine the points in the usual *xy*-plane where the circle $x^2 + y^2 = 16$ and the parabola $y = x^2 - 4$ intersect.

Explanation:

Using substitution: $(y + 4) + y^2 = 16 \Rightarrow y = -4 \text{ or } 3$

If $y = -4$, then $x = 0$, but if $y = 3$, then $x = \pm\sqrt{7}$.

Thus, the circle and parabola intersect at the points: $(0, -4)$, $(\sqrt{7}, 3)$, $(-\sqrt{7}, 3)$.

Answer: $(0, -4)$, $(\sqrt{7}, 3)$, $(-\sqrt{7}, 3)$.

14. Simplify completely: $\left(\dfrac{\log_5 125}{\log_5 9}\right)\left(\dfrac{\log_5 81}{\log_2 \sqrt[5]{5}}\right)\left(\dfrac{\log_2 125}{\log_b b^3}\right)$

Explanation: Rewrite the expression as:

$$\left(\frac{\log_5 5^3}{\log_5 9}\right)\left(\frac{\log_5 9^2}{\log_2 5^{\frac{1}{5}}}\right)\left(\frac{\log_2 5^3}{\log_b b^3}\right) \Leftrightarrow \frac{3\bullet 2\cancel{\log_5 9}\bullet 3\cancel{\log_2 5}}{\cancel{\log_5 9}\bullet \frac{1}{5}\cancel{\log_2 5}\bullet 3\log_b b} \Leftrightarrow \frac{3\bullet 2\bullet 3}{\frac{1}{5}\bullet 3} \Leftrightarrow 30$$

Answer: 30

15. Find all real solutions of the equation: $2\left(x^2-3\right)^2+3\left(x^2-3\right)=4x^2-12$

Explanation:

1st factor the right side of the equation: $2\left(x^2-3\right)^2+3\left(x^2-3\right)=4\left(x^2-3\right)$

2nd let $z = (x^2 - 3)$, then solve: $2z^2 + 3z = 4z \Rightarrow z=0 \;\; or \;\; \frac{1}{2}$

3rd un-substitute and solve: $x^2-3=0 \;\; or \;\; x^2-3=\frac{1}{2}$

Answer: $\left\{\pm\sqrt{3}, \pm\frac{\sqrt{14}}{2}\right\}$

16. Five pennies are flipped independently of one another. Assume that each penny is fair; a ‘head’ or a ‘tail’ is equally likely. What is the probability that at least one of the pennies will come up ‘heads’?

Explanation: $P(\text{at least one } H)=1-P(\text{all } T)=1-\left(\frac{1}{2}\right)^5=1-\frac{1}{32}=\frac{31}{32}$

Answer: $\frac{31}{32}$

17. If $f(x)=5+x-x^2$, find $\dfrac{f(a+h)-f(a)}{h}$ in completely simplified form. Assume $h\neq 0$.

Explanation:

$$\frac{5+\left(a+h\right)-\left(a+h\right)^2-(5+a-a^2)}{h} \Leftrightarrow \frac{h-2ah-h^2}{h} \Leftrightarrow 1-2a-h$$

Answer: $1-2a-h$

18. Find all real solutions of the inequality: $\frac{x^2}{2x-3} > 3$

Explanation:
Solve the equation:

$$\frac{x^2}{2x-3} = 3 \Rightarrow x^2 = 3(2x-3) \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$$

But 3 (along with 3/2, because $2x-3 \neq 0$) are not solutions, merely 'critical values.' We must test the intervals created along the number line:

$x < 3/2$	$3/2 < x < 3$	$x > 3$
Test 0	Test 2	Test 4
Result: 0 > 3 False	Result: 4 > 3 True	Result: 16/5 > 3 True

The solution set consists of the intervals of the chart that results in a 'true' value:

Answer: $\left\{x \in \mathbb{R} \,\middle|\, x > \frac{3}{2}, x \neq 3\right\}$, or $\left(\frac{3}{2}, 3\right) \cup (3, \infty)$

19. Solve for *y* in terms of *x*: $9x^2 - 4y^2 = 25 - 16y$

Explanation:
Basic plan: get all '*y*'s on one side of the =, then complete the square on the '*y*'s.

$$9x^2 - 25 = 4y^2 - 16y \Leftrightarrow \frac{9x^2-25}{4} = y^2 - 4y \Leftrightarrow \frac{9x^2-25}{4} + 4 = y^2 - 4y + 4 \Leftrightarrow$$

$$\frac{9x^2-25}{4} + 4 = (y-2)^2 \Leftrightarrow y - 2 = \pm\sqrt{\frac{9x^2-9}{4}} \Leftrightarrow y = 2 \pm \frac{3}{2}\sqrt{x^2-1}$$

Answer: $y = 2 \pm \frac{3}{2}\sqrt{x^2-1}$, or $y = \frac{4 \pm 3\sqrt{x^2-1}}{2}$

20. What digit is in the ones place of 7^{2012}?

Explanation: We will consider some powers of 7 and wait until a last digit is repeated; then, we automatically get a cycling pattern. Observe that only the ones digits of these powers of 7 are relevant.

n	Last digit of 7^n	n	Last digit of 7^n
1	7	6	9
2	9	7	3
3	3	**8**	**1**
4	**1**	9	7
5	7	…	…

The ones digits will cycle in sequences of length 4. The last digits of $7^4, 7^8, 7^{12}, \ldots, 7^{2012}$ will all be '1's.

Answer: 1